Asymptotic Formulas for the Heat Kernels of Space and Time Fractional Equations

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Content

Mainly based on a joint work with Rene Schilling (TU Dresden, Germany): arXiv:1803.11435

1. Background and motivation
2. Main results
3. Basic idea of the proofs

- Recall the classical heat equation on $\mathbb{R}^d$

$$\frac{\partial u}{\partial t} = \Delta u.$$

- Its fundamental solution is given by the Gauss kernal

$$p(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left[-\frac{|x-y|^2}{4t}\right], \quad t>0,\ x,y \in \mathbb{R}^d.$$

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Space-fractional equation

- The non-local heat equation ($0 < \beta < 1$)

$$\frac{\partial u}{\partial t} = -(-\Delta)^{\beta} u.$$

- 2β-stable Lévy process W_{S_t}.

  Here S_t is an independent β-stable subordinator with

$$\mathbb{E}\,\mathrm{e}^{-rS_t} = \mathrm{e}^{-tr^{\beta}}, \quad r > 0, t \geq 0.$$

- By independence, the heat kernel is

$$\mathbb{E}\,p(S_t, x, y) = \int_0^{\infty} p(s, x, y)\,\mathbb{P}(S_t \in \mathrm{d}s).$$

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Time-fractional heat equation $(0 < \beta < 1)$

$$\frac{\partial^\beta u}{\partial t^\beta} = \Delta u.$$

Here $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo derivative:

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\beta}\big(f - f(0)\big)(t).$$

The Riemann-Liouville integral operator:

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} \,\mathrm{d}s.$$

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$$S_t^{-1} := \inf \{ s \geq 0 \, : \, S_s > t \}, \quad t \geq 0.$$

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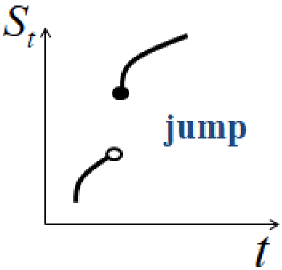
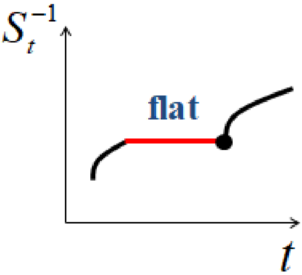
Speed under time-change





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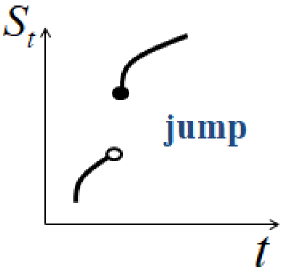
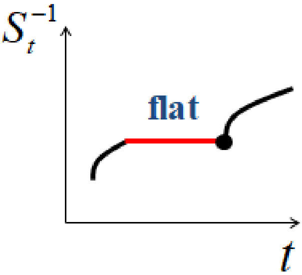
- $W_{S_t^{-1}} \overset{\mathrm{d}}{=} t^{\beta/2} W_{S_1^{-1}}$ $\qquad$ (subdiffusion, slow diffusion)





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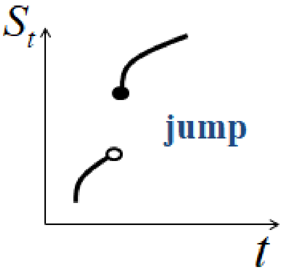

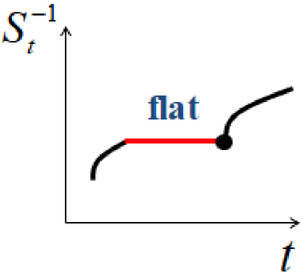

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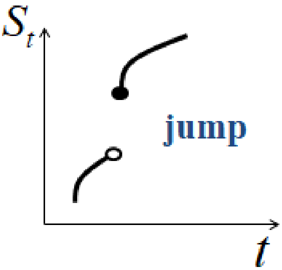

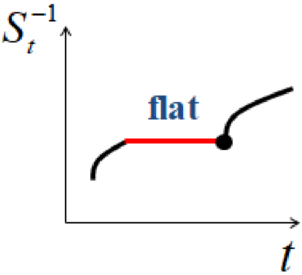

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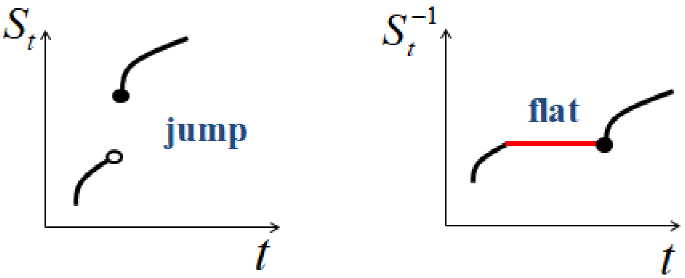

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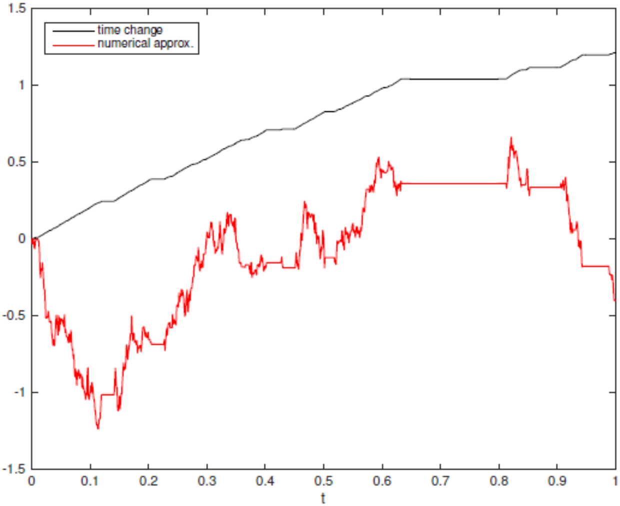
A sample path (by Jin-Kobayashi, 2019)



Black: S_t^{-1} **Red:** $W_{S_t^{-1}}$

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PDE vs probability

PDE	Heat Kernel	Process
$\partial_t u = \Delta u$	$p(t, x, y)$	W_t
$\partial_t u = -(-\Delta)^\beta u$	$\mathbb{E}\, p(S_t, x, y)$	W_{S_t}
$\partial_t^\beta u = \Delta u$	$\mathbb{E}\, p(S_t^{-1}, x, y)$	$W_{S_t^{-1}}$
$\partial_t^\beta u = -(-\Delta)^\gamma u$	$\mathbb{E}\, p(T_{S_t^{-1}}, x, y)$	$W_{T_{S_t^{-1}}}$

Heat kernel of $\frac{\partial u}{\partial t} = -(-\Delta)^\beta u$

- $p(t,x,y)$ is the Gauss heat kernel, and S_t is a β-stable subordinator.

- When $\beta = 1/2$,

$$\mathbb{E}\, p(S_t,x,y) = \frac{c(d)}{t^d}\left(1+\frac{|x-y|^2}{t^2}\right)^{-(d+1)/2}.$$

- If $\beta \neq 1/2$, NO explicit formula for $\mathbb{E}\, p(S_t,x,y)$!!!

- A natural question: **asymptotic formula?**

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Motivation 1

- It is known that as $|x-y|^2 t^{-1/\beta} \to \infty$,

$$\mathbb{E}p(S_t, x, y) \sim \frac{c(d,\beta)t}{\left(|x-y|^2 + t^{1/\beta}\right)^{(d+2\beta)/2}}.$$

Pólya, 1923, $d=1$

Blumenthal and Getoor, 1960, general $d \geq 1$

Tool: Bessel function

A. Bendikov, 1994, a quite elegant proof

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Motivation 2: heat kernel under inverse subordination

- $p(t, x, y)$ is the Gauss heat kernel, and S_t is a β-stable subordinator.
- It seems impossible to expect explicit formula for $\mathbb{E}\, p(S_t^{-1}, x, y)$, i.e. the heat kernel of $W_{S_t^{-1}}$ (**Note:** non-Markovian).
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Motivation 3: fractional in both space and time

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Main result

Consider the fundamental solution $p(t,x,y)$ to

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Theorem (D.-Schilling, 2019+)

(1) As $|x-y|^{-2\gamma/\beta} t \to \infty$, $p(t,x,y)$ is equivalent to

$$\begin{cases} \dfrac{\Gamma\left(\frac{1}{2\gamma}\right)\Gamma\left(1-\frac{1}{2\gamma}\right)}{2\pi\gamma\Gamma\left(1-\frac{\beta}{2\gamma}\right)} t^{-\beta/(2\gamma)}, & d=1 \ \& \ \gamma \in \left(\frac{1}{2},1\right), \\ \dfrac{\beta}{\pi\Gamma(1-\beta)} t^{-\beta} \log\left[|x-y|^{-1/\beta} t\right], & d=1 \ \& \ \gamma = \frac{1}{2}, \\ \dfrac{2\gamma\Gamma\left(\frac{d-2\gamma}{2}\right)}{2^{1+2\gamma}\pi^{d/2}\Gamma(1-\beta)\Gamma(1+\gamma)} |x-y|^{2\gamma-d} t^{-\beta}, & d > 2\gamma \ \& \ \gamma \in (0,1). \end{cases}$$

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(1) As $|x-y|^{-2\gamma/\beta} t \to \infty$, $p(t,x,y)$ is equivalent to

$$\begin{cases} \dfrac{\Gamma\left(\frac{1}{2\gamma}\right)\Gamma\left(1-\frac{1}{2\gamma}\right)}{2\pi\gamma\Gamma\left(1-\frac{\beta}{2\gamma}\right)} t^{-\beta/(2\gamma)}, & d=1 \ \& \ \gamma \in (\frac{1}{2},1), \\ \dfrac{\beta}{\pi\Gamma(1-\beta)} t^{-\beta} \log\left[|x-y|^{-1/\beta} t\right], & d=1 \ \& \ \gamma = \frac{1}{2}, \\ \dfrac{2\gamma\Gamma\left(\frac{d-2\gamma}{2}\right)}{2^{1+2\gamma}\pi^{d/2}\Gamma(1-\beta)\Gamma(1+\gamma)} |x-y|^{2\gamma-d} t^{-\beta}, & d > 2\gamma \ \& \ \gamma \in (0,1). \end{cases}$$

Main result

Consider the fundamental solution $p(t,x,y)$ to

$$\frac{\partial^\beta u}{\partial t^\beta} = -(-\Delta)^\gamma u.$$

Theorem (cont.)

(2) As $|x-y|^{-2\gamma/\beta} t \to 0$,

$$p(t,x,y) \sim \frac{\gamma 4^\gamma \Gamma\left(\frac{d}{2}+\gamma\right)}{\pi^{d/2}\Gamma(1-\gamma)\beta\Gamma(\beta)} |x-y|^{-d-2\gamma} t^\beta.$$

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General framework

- Let $p(t,x,y)$ be a heat kernel on a metric space (M,ρ)

$$p(t,x,y)=\frac{C_1}{t^{d/\alpha}}F\left(C_2\frac{\rho(x,y)}{t^{1/\alpha}}\right),\quad t>0,\, x,y\in M,$$

where $d,\alpha,C_1,C_2>0$ and $F:[0,\infty)\to(0,\infty)$ is $\downarrow$.

- Typical examples of F are

$$F(r)=\exp\left[-r^{\alpha/(\alpha-1)}\right]\quad \text{with some } \alpha\geq 2, \qquad (\bigstar)$$

$$F(r)=(1+r^2)^{-(d+\alpha)/2}\quad \text{with some } \alpha>0. \qquad (\bigstar\bigstar)$$

- Indeed, under some conditions, Grigor'yan-Kumagai (2008),

$$p(t,x,y)\asymp\frac{C_1}{t^{d/\alpha}}F\left(C_2\frac{\rho(x,y)}{t^{1/\alpha}}\right),$$

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Related fractional SPDE (M. Foodun, E. Nane, R. Sun, $\cdots$):

$$\frac{\partial^\beta u}{\partial t^\beta} = -(-\Delta)^\gamma u + \sigma(u)\dot{W}(t,x),$$

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$$p(t,x,y)=\frac{1}{(4\pi t)^{d/2}}\exp\left[-\frac{|x-y|^2}{4t}\right].$$

- Poisson kernel

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$$p(t,x,y)=\frac{C_1}{t^{d/\alpha}}F\left(C_2\frac{\rho(x,y)}{t^{1/\alpha}}\right),\quad t>0,\ x,y\in M.$$

- Let X_t be the Markov process on M associated with $p(t,x,y)$, and denote by $\mathcal{L}$ the generator.
- Let S_t be an independent β-stable subordinator.
- Our aim: $\mathbb{E}\,p(S_t,x,y)$ and $\mathbb{E}\,p(S_t^{-1},x,y)$.
- They are the fundamental solutions to

  $$\frac{\partial u}{\partial t}=-(-\mathcal{L})^\beta u$$

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$$\mathbb{E}\, p(S_t,x,y) \sim \frac{C_1\alpha\beta}{\Gamma(1-\beta)} C_2^{-d-\alpha\beta} \int_0^\infty s^{d+\alpha\beta-1} F(s)\,\mathrm{d}s \cdot \rho(x,y)^{-d-\alpha\beta} t.$$

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$$\mathbb{E}\, p(S_t, x, y) = \int_0^\infty p(s, x, y)\, \mathbb{P}(S_t \in \mathrm{d}s).$$

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♠ Asymptotic for the density of stable subordinator:

$$\frac{\mathbb{P}(S_1 \in \mathrm{d}s)}{\mathrm{d}s} \sim \begin{cases} C_1(\beta)\, s^{-\frac{2-\beta}{2(1-\beta)}} \exp\left[-C_2(\beta)\, s^{-\frac{\beta}{1-\beta}}\right], & \text{as } s \to 0, \\ \frac{\beta}{\Gamma(1-\beta)} s^{-\beta-1}, & \text{as } s \to \infty. \end{cases}$$

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Technique from asymptotic analysis

Asymptotic formula (Laplace method/ Tauberian theorem)

If $h \geq 0$ has a unique local minimum point at $r_0 \in (0, \infty)$ and $h''(r_0) > 0$,

$$\int_0^\infty \mathrm{e}^{-Ch(r)}\, \mathrm{d}r \sim \mathrm{e}^{-Ch(r_0)} \sqrt{\frac{2\pi}{Ch''(r_0)}} \qquad \text{as } C \to \infty.$$

Possible extension/ Future work

$$\frac{\partial u}{\partial t} = \Delta u \quad \longrightarrow \quad \frac{\partial^\beta u}{\partial t^\beta} = -(-\Delta)^\gamma u \qquad \surd \quad \text{Finished}$$

$$\longrightarrow \quad \frac{\partial^\beta u}{\partial t^\beta} = -\phi(-\Delta)u \quad \ldots ?$$

where ϕ is the so-called Bernstein function.

♠ Asymptotic expansion?

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Thanks for Your Attention!